

Integral equation for electrostatic waves generated by a point source in a spatially homogeneous magnetized plasma

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Abstract

The electric field generated by a time varying point charge in a three-dimensional, unbounded, spatially homogeneous plasma with a uniform background magnetic field and a uniform (static) flow velocity is studied in the electrostatic approximation which is often valid in the near field. For plasmas characterized by Maxwell distribution functions with isotropic temperatures, the linearized Vlasov-Poisson equations may be formulated in terms of an equivalent integral equation in the time domain. The kernel of the integral equation has a relatively simple mathematical form consisting of elementary functions such as exponential and trigonometric functions (sines and cosines), and contains no infinite sums of Bessel functions. Consequently, the integral equation is amenable to numerical solutions and may be useful for the study of the impulse response of magnetized plasmas and, more generally, the response to arbitrary waveforms.

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I. INTRODUCTION

The study of plasma waves generated by an oscillating point charge or point dipole—fixed in space but oscillating in time—in a spatially homogeneous hot magnetized plasma is an important model problem that has been investigated both theoretically and experimentally since the 1960's [1–39]. Following the pioneering work of Landau in 1946 [40], the fully kinetic self-consistent field approach described by the coupled Vlasov and Maxwell equations has been used to solve and study such systems, mainly through Fourier and Laplace transform analysis. Experience with these techniques shows that even in the simplest spatially homogeneous systems, the presence of ambient magnetic fields usually complicates the mathematical analysis significantly.

Here we show that in the electrostatic approximation and for particle distribution functions with isotropic temperatures, an alternative approach based on solving an integral equation in the time domain provides a relatively simple mathematical formulation that contains no Bessel function series. Similar formulations may be possible for gyrotropic distributions with anisotropic temperatures, but these are not considered here. To the author's knowledge, the integral equation presented here for the field of a point source in a magnetized plasma has not appeared before in the literature.

Solutions of the linearized Vlasov-Maxwell or Vlasov-Poisson equations for specified charge and current distributions are of interest in many applications where the steady state solution for time harmonic forcing and the impulse response corresponding to a delta function forcing (in time) are usually of special interest. In the electrostatic approximation and in the presence of a constant background magnetic field, the steady state solution for a point source can usually be expressed in terms of an infinite series of modified Bessel functions [19, 30, 34, 41–43]. Here, it is shown that the solutions for a point source with any prescribed time dependence can be derived from an integral equation in which the kernel is free of such Bessel function series. Since the integral equation formulation is amenable to numerical calculations, it can be used to study the impulse response of magnetized plasmas as well as the response to other arbitrary waveforms through direct calculation. In that respect, it has more flexibility and some other advantages compared to the Laplace transform approach.

II. STATEMENT OF THE PROBLEM

Consider a spatially unbounded and homogeneous plasma with a background magnetic field $\mathbf{B}_0 = B_0 \hat{e}_z$, $B_0 > 0$, and an externally imposed point charge with charge density $q(t)\delta(\mathbf{x})$, where $q(t)$ is a given function of time, $\delta(\mathbf{x}) = \delta(x)\delta(y)\delta(z)$ is the three dimensional delta function, $\delta(x)$ is the (one dimensional) Dirac delta function, and (x, y, z) are the usual orthogonal cartesian coordinates in three dimensional space with corresponding unit vectors $\hat{e}_x, \hat{e}_y, \hat{e}_z$. The function $q(t)$ is assumed to vanish for $t \leq 0$ and is continuous for $t \geq 0$. Hence, by causality, the plasma response will vanish at $t = 0$ or, in other words, the initial conditions on the perturbed fields and the perturbed distribution functions all vanish at $t = 0$. For $t > 0$, the point charge excites plasma waves which propagate away from the origin and the problem is to compute the fields produced by this point source.

At equilibrium, the plasma is charge neutral and current free which is expressed by the relations

$$\sum_s n_s q_s = 0 \quad \text{and} \quad \sum_s n_s q_s \mathbf{V}_s = 0 \quad (1)$$

where n_s , q_s and \mathbf{V}_s are the equilibrium number density, charge, and bulk flow velocity of particle species s , respectively. The equilibrium distribution functions are all assumed to be non-relativistic convected Maxwell distributions with isotropic temperatures of the form

$$f_{0s}(v) = \frac{1}{(\pi v_s^2)^{3/2}} \exp \left[-\frac{|\mathbf{v} - \mathbf{V}_s|^2}{v_s^2} \right], \quad (2)$$

where $v_s = (2k_B T_s / m_s)^{1/2}$ is the thermal speed, k_B is Boltzmann's constant, m_s is the particle mass, and T_s is the kinetic temperature. Note, however, that the analysis below can readily be generalized to other distribution functions with isotropic temperatures. The Maxwell distribution (2) has the convenient property

$$\nabla_{\mathbf{v}} f_{0s} = -\frac{2}{v_s^2} (\mathbf{v} - \mathbf{V}_s) f_{0s}(v) \quad (3)$$

that shall be used below.

To linearize the Vlasov-Poisson equations, the first step is to specify the equilibrium state. In the presence of plasma flow in an arbitrary direction, the equilibrium solution of the collisionless Vlasov equation must satisfy $(\mathbf{E}_0 + \mathbf{v} \times \mathbf{B}_0) \cdot (\mathbf{v} - \mathbf{V}_s) = 0$ which implies the existence of a static electric field $\mathbf{E}_0 = -\mathbf{V}_s \times \mathbf{B}_0$. For the solution to be self consistent, the component of the velocity transverse to \mathbf{B}_0 must be the same for each species, that is, $\mathbf{V}_s \times \mathbf{B}_0$ must be the same for each

s. The equilibrium state consisting of \mathbf{E}_0 , \mathbf{B}_0 , and $f_0(\mathbf{v})$ can now be used to linearize the Vlasov equation. To simplify the presentation, the ion response shall be neglected from now on and only the electron contribution shall be considered; because the susceptibilities of the different particle species are additive, the ion response can easily be taken into account as shown at the end of the derivation.

The perturbed electric field \mathbf{E}_1 is assumed to be electrostatic meaning that $\mathbf{E}_1 = -\nabla\phi$, where $\phi(\mathbf{x}, t)$ is the electrostatic potential, an approximation that is valid in the near field [12, 19], at least over some range of physical parameters. The precise range of validity of the electrostatic approximation for the problem under consideration is unknown and shall not be studied here. However, for time harmonic forcing, the steady state solution can be computed numerically both with and without the electrostatic approximation, thus providing a means of comparison.

In terms of the velocity variable $\mathbf{u} = \mathbf{v} - \mathbf{V}$, where the subscript on $\mathbf{V} = \mathbf{V}_e$ is omitted for convenience, the linearized Vlasov equation takes the form

$$\frac{\partial f_1}{\partial t} + (\mathbf{u} + \mathbf{V}) \cdot \nabla f_1 - \frac{e}{m_e} (\mathbf{u} \times \mathbf{B}_0) \cdot \nabla_{\mathbf{u}} f_1 = -\frac{e}{m_e} \nabla \phi \cdot \nabla_{\mathbf{u}} f_0, \quad (4)$$

where $f_1(\mathbf{x}, \mathbf{u}, t)$ is the perturbed distribution function defined such that the complete distribution function is $n_e(f_0 + f_1)$ with $f_0(\mathbf{u})$ normalized to unity and $e > 0$ the electronic charge. Express the velocity vector in cylindrical coordinates as

$$\mathbf{u} = u_{\perp} \cos(\varphi) \hat{\mathbf{e}}_x + u_{\perp} \sin(\varphi) \hat{\mathbf{e}}_y + u_{\parallel} \hat{\mathbf{e}}_z. \quad (5)$$

After Fourier transformation with respect to the spatial variables, the Vlasov equation becomes

$$\frac{\partial \tilde{f}_1}{\partial t} + i\mathbf{k} \cdot (\mathbf{u} + \mathbf{V}) \tilde{f}_1 - \Omega_e \frac{\partial \tilde{f}_1}{\partial \varphi} = \frac{2e\tilde{\phi}}{v_e^2 m_e} i\mathbf{k} \cdot \mathbf{u} f_0(u), \quad (6)$$

where $\tilde{f}_1(\mathbf{k}, \mathbf{u}, t)$ and $\tilde{\phi}(\mathbf{k}, t)$ are the respective Fourier transforms of $f_1(\mathbf{x}, \mathbf{u}, t)$ and $\phi(\mathbf{x}, t)$, and $\Omega_e = -eB_0/m_e$ is the signed electron cyclotron frequency ($\Omega_e < 0$). The problem is to solve the Vlasov equation (6) for $t \geq 0$ together with Poisson's equation

$$k^2 \tilde{\phi}(\mathbf{k}, t) = -\frac{n_e e}{\epsilon_0} \int \tilde{f}_1(\mathbf{k}, \mathbf{u}, t) d\mathbf{u} + \frac{q(t)}{\epsilon_0} \quad (7)$$

subject to the initial conditions $\tilde{f}_1(\mathbf{k}, \mathbf{u}, 0) = 0$ and $\tilde{\phi}(\mathbf{k}, 0) = 0$, where $q(t)$ is a continuous function such that $q(0) = 0$. Otherwise, the forcing function $q(t)$ is arbitrary.

III. FORMULATION AS AN INTEGRAL EQUATION

The goal is to reduce the initial value problem consisting of the Vlasov-Poisson equations (6) and (7) to an equivalent integral equation. The result is as follows. The potential $\tilde{\phi}(\mathbf{k}, t)$ which solves the Vlasov-Poisson system is also the solution of the Volterra integral equation of the second kind

$$\tilde{\phi}(\mathbf{k}, t) + \omega_{pe}^2 \int_0^t \Gamma(\mathbf{k}, t - \tau) \tilde{\phi}(\mathbf{k}, \tau) d\tau = \frac{q(t)}{\epsilon_0 k^2}, \quad k \neq 0, \quad (8)$$

with kernel

$$\begin{aligned} \Gamma(\mathbf{k}, \tau) = \tau \left[\frac{k_{\perp}^2}{k^2} \cdot \frac{\sin(\Omega_e \tau)}{\Omega_e \tau} + \frac{k_{\parallel}^2}{k^2} \right] \\ \times \exp \left\{ -\frac{k_{\perp}^2 v_e^2}{\Omega_e^2} \sin^2 \left(\frac{\Omega_e \tau}{2} \right) - \left(\frac{k_{\parallel} v_e \tau}{2} \right)^2 - i\mathbf{k} \cdot \mathbf{V} \tau \right\}. \end{aligned} \quad (9)$$

Note that the kernel is a convolution kernel as is required for a linear time invariant system. Remarkably, the kernel (9) does not contain the infinite sums of Bessel functions that occur in steady state solutions with time harmonic forcing. The terms containing Ω_e represent the magnetic field effects. In the limit as $\Omega_e \rightarrow 0$, the function (9) reduces to the correct kernel for a point charge in an unmagnetized plasma

$$\Gamma(\mathbf{k}, \tau) = \tau \exp \left\{ -\left(\frac{k v_e \tau}{2} \right)^2 - i\mathbf{k} \cdot \mathbf{V} \tau \right\}, \quad (10)$$

where $k = (k_{\perp}^2 + k_{\parallel}^2)^{1/2}$. Because the kernel (9) is smooth (continuously differentiable) and consists of compositions of algebraic, exponential and trigonometric functions (sines and cosines), it is convenient for purposes of numerical calculation. When only the electron response is considered there are two inherent timescales which must be resolved, the electron cyclotron period $2\pi/\omega_{ce}$ and the inverse plasma frequency $2\pi/\omega_{pe}$.

Integral equation formulations of the Vlasov-Poisson system in unmagnetized plasmas are well known [44–52]. However, to the authors' knowledge, the integral equation formulation for a magnetized plasma given by (8) and (9) is new. Of course, the literature is vast, spanning more than 50 years, and it is possible that this result may have been derived previously in work unknown to the present author. The remainder of this paper is devoted to a derivation of the integral equation (8) and the kernel (9).

IV. SOLUTION OF THE VLASOV EQUATION

Similar to (5), let

$$\mathbf{k} = k_{\perp} \cos(\theta) \hat{\mathbf{e}}_x + k_{\perp} \sin(\theta) \hat{\mathbf{e}}_y + k_{\parallel} \hat{\mathbf{e}}_z. \quad (11)$$

Then

$$\mathbf{k} \cdot \mathbf{u} = k_{\perp} u_{\perp} \cos(\varphi - \theta) + k_{\parallel} u_{\parallel}. \quad (12)$$

The substitution

$$\tilde{f}_1 = \psi \exp \left\{ + i \frac{k_{\perp} u_{\perp}}{\Omega_e} \sin(\varphi - \theta) - i(k_{\parallel} u_{\parallel} + \mathbf{k} \cdot \mathbf{V})t \right\} \quad (13)$$

brings the Vlasov equation (6) into the form

$$\frac{\partial \psi}{\partial t} - \Omega_e \frac{\partial \psi}{\partial \varphi} = \frac{2e}{v_e^2 m_e} \tilde{\phi}(\mathbf{k}, t) f_0(u) i \mathbf{k} \cdot \mathbf{u} \exp \left\{ - i \frac{k_{\perp} u_{\perp}}{\Omega_e} \sin(\varphi - \theta) + i(k_{\parallel} u_{\parallel} + \mathbf{k} \cdot \mathbf{V})t \right\}. \quad (14)$$

Consider the transformation from independent variables t and φ to a new set of independent variables η and ξ defined by

$$\begin{cases} \eta = \frac{1}{2}(\varphi + \Omega_e t) \\ \xi = \frac{1}{2}(\varphi - \Omega_e t). \end{cases} \quad (15)$$

It follows that

$$\frac{\partial \psi}{\partial t} - \Omega_e \frac{\partial \psi}{\partial \varphi} = -\Omega_e \frac{\partial \psi}{\partial \xi} \quad (16)$$

and, therefore, the solution of (14) is obtained by integrating both sides of that equation with respect to ξ while holding η fixed. Because $t = 0$ is equivalent to $\xi = \eta$, the integration is performed from the lower limit $\xi' = \eta$ to $\xi' = \xi$, where ξ' is the variable of integration. To express the right-hand side of (14) in terms of η and ξ , substitute $\varphi = \eta + \xi$ and $t = (\eta - \xi)/\Omega_e$ so, for example,

$$\mathbf{k} \cdot \mathbf{u} = k_{\perp} u_{\perp} \cos(\eta + \xi - \theta) + k_{\parallel} u_{\parallel}. \quad (17)$$

The integration of (14) implies

$$\begin{aligned} \psi(\eta, \xi) - \psi(\eta, \eta) = & -\frac{2e}{v_e^2 m_e} \int_{\eta}^{\xi} \frac{d\xi'}{\Omega_e} \tilde{\phi}\left(\mathbf{k}, \frac{\eta - \xi'}{\Omega_e}\right) f_0(u) e^{-i\mathbf{k} \cdot \mathbf{V}(\eta - \xi')/\Omega_e} \\ & \times i[k_{\perp} u_{\perp} \cos(\eta + \xi' - \theta) + k_{\parallel} u_{\parallel}] \exp \left\{ - i \frac{k_{\perp} u_{\perp} \sin(\eta + \xi' - \theta) + k_{\parallel} u_{\parallel}(\xi' - \eta)}{\Omega_e} \right\}. \end{aligned} \quad (18)$$

By definition, $\psi(\eta, \eta)$ is equal to $\psi(t = 0)$ which is equal to zero by virtue of the vanishing initial condition on f_1 . The change of variable $\tau = (\eta - \xi')/\Omega_e$, where η is a constant, yields

$$\psi(\eta, \xi) = \frac{2e}{v_e^2 m_e} \int_0^t d\tau \tilde{\phi}(\mathbf{k}, \tau) f_0(u) i [k_\perp u_\perp \cos(2\eta - \theta - \Omega_e \tau) + k_\parallel u_\parallel] \exp \left\{ -i \frac{k_\perp u_\perp}{\Omega_e} \sin(2\eta - \theta - \Omega_e \tau) + i(k_\parallel u_\parallel + \mathbf{k} \cdot \mathbf{V})\tau \right\}. \quad (19)$$

Because η is held constant inside the integrand, if one makes the substitution $\eta = (\varphi + \Omega_e t)/2$ inside the integrand with both φ and t held constant during the integration, then the result of the integration is the same as if the substitution $\eta = (\varphi + \Omega_e t)/2$ were made after the integration. Therefore, the preceding equation is equivalent to

$$\psi(\mathbf{k}, \mathbf{u}, t) = \frac{2e}{v_e^2 m_e} \int_0^t d\tau \tilde{\phi}(\mathbf{k}, \tau) f_0(u) i \{ k_\perp u_\perp \cos[\varphi - \theta + \Omega_e(t - \tau)] + k_\parallel u_\parallel \} \exp \left\{ -i \frac{k_\perp u_\perp}{\Omega_e} \sin[\varphi - \theta + \Omega_e(t - \tau)] + i(k_\parallel u_\parallel + \mathbf{k} \cdot \mathbf{V})\tau \right\}. \quad (20)$$

It can be verified by direct calculation that this is the solution of the partial differential equation (14). Thus, from (13), the solution of the Vlasov equation (6) is

$$\tilde{f}_1(\mathbf{k}, \mathbf{u}, t) = \frac{2e}{v_e^2 m_e} \int_0^t d\tau \tilde{\phi}(\mathbf{k}, \tau) f_0(u) i \{ k_\perp u_\perp \cos[\varphi - \theta + \Omega_e(t - \tau)] + k_\parallel u_\parallel \} \exp \left[-i \frac{k_\perp u_\perp}{\Omega_e} \left\{ \sin[\varphi - \theta + \Omega_e(t - \tau)] - \sin(\varphi - \theta) \right\} - i(k_\parallel u_\parallel + \mathbf{k} \cdot \mathbf{V})(t - \tau) \right] \quad (21)$$

or, equivalently,

$$\tilde{f}_1(\mathbf{k}, \mathbf{u}, t) = \frac{2e}{v_e^2 m_e} \int_0^t d\tau \tilde{\phi}(\mathbf{k}, t - \tau) f_0(u) i [k_\perp u_\perp \cos(\varphi - \theta + \Omega_e \tau) + k_\parallel u_\parallel] \exp \left\{ -i \frac{k_\perp u_\perp}{\Omega_e} [\sin(\varphi - \theta + \Omega_e \tau) - \sin(\varphi - \theta)] - i(k_\parallel u_\parallel + \mathbf{k} \cdot \mathbf{V})\tau \right\} \quad (22)$$

This result is now inserted into Poisson's equation (7) and the integration over velocity space is performed.

V. VELOCITY SPACE INTEGRALS

The integration of \tilde{f}_1 with respect to \mathbf{u} is performed in cylindrical coordinates. To facilitate the calculation, it is expedient to write (22) in the form

$$\tilde{f}_1(\mathbf{k}, \mathbf{u}, t) = -\frac{2e}{v_e^2 m_e} \int_0^t d\tau \tilde{\phi}(\mathbf{k}, t - \tau) f_0(u) e^{-i\mathbf{k} \cdot \mathbf{V}\tau} \times \frac{\partial}{\partial \tau} \exp \left\{ -i \frac{k_\perp u_\perp}{\Omega_e} [\sin(\varphi - \theta + \Omega_e \tau) - \sin(\varphi - \theta)] - i k_\parallel u_\parallel \tau \right\} \quad (23)$$

The integral with respect to φ is readily computed using the Bessel function series

$$e^{iz \sin \phi} = \sum_{n=-\infty}^{\infty} e^{in\phi} J_n(z), \quad (24)$$

where $J_n(z)$ is the Bessel function of the first kind of order n and argument z , and z is an arbitrary complex number [53]. It follows from this series expansion that

$$\exp \left[i \frac{k_{\perp} u_{\perp}}{\Omega_e} \sin(\varphi - \theta) \right] = \sum_{m=-\infty}^{\infty} e^{im(\varphi - \theta)} J_m \left(\frac{k_{\perp} u_{\perp}}{\Omega_e} \right) \quad (25)$$

and

$$\exp \left[i \frac{k_{\perp} u_{\perp}}{\Omega_e} \sin(\varphi - \theta + \Omega_e \tau) \right] = \sum_{n=-\infty}^{\infty} e^{in(\varphi - \theta + \Omega_e \tau)} J_n \left(\frac{k_{\perp} u_{\perp}}{\Omega_e} \right). \quad (26)$$

The orthogonality relations for the complex exponentials then yield

$$\begin{aligned} \int_0^{2\pi} d\varphi \exp \left\{ -i \frac{k_{\perp} u_{\perp}}{\Omega_e} [\sin(\varphi - \theta + \Omega_e \tau) - \sin(\varphi - \theta)] \right\} \\ = 2\pi \sum_{n=-\infty}^{\infty} \left[J_n \left(\frac{k_{\perp} u_{\perp}}{\Omega_e} \right) \right]^2 e^{in\Omega_e \tau}. \end{aligned} \quad (27)$$

The series on the right-hand side is a special case of the Fourier series

$$\sum_{n=-\infty}^{\infty} [J_n(z)]^2 e^{in\phi} = J_0 \left(2z \sin \frac{\phi}{2} \right) \quad (28)$$

with Fourier coefficients given by [54, page 48, equation 13]

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} J_0 \left(2z \sin \frac{\phi}{2} \right) e^{-in\phi} d\phi = [J_n(z)]^2. \quad (29)$$

Hence,

$$\begin{aligned} \int_0^{2\pi} d\varphi \exp \left\{ -i \frac{k_{\perp} u_{\perp}}{\Omega_e} [\sin(\varphi - \theta + \Omega_e \tau) - \sin(\varphi - \theta)] \right\} \\ = 2\pi J_0 \left(\frac{2k_{\perp} u_{\perp}}{\Omega_e} \sin \frac{\Omega_e \tau}{2} \right) \end{aligned} \quad (30)$$

and, from (23),

$$\begin{aligned} \int_0^{2\pi} d\varphi \tilde{f}_1(\mathbf{k}, \mathbf{u}, t) = -\frac{4\pi e}{v_e^2 m_e} \int_0^t d\tau \tilde{\phi}(\mathbf{k}, t - \tau) f_0(u) e^{-i\mathbf{k} \cdot \mathbf{V} \tau} \\ \times \frac{\partial}{\partial \tau} \left\{ J_0 \left(\frac{2k_{\perp} u_{\perp}}{\Omega_e} \sin \frac{\Omega_e \tau}{2} \right) e^{-ik_{\parallel} u_{\parallel} \tau} \right\} \end{aligned} \quad (31)$$

It remains to perform the integration with respect to u_\perp and u_\parallel . The integration with respect to u_\perp is accomplished by means of Weber's first exponential integral [55, page 393]

$$\frac{1}{\pi v_e^2} \int_0^\infty u_\perp e^{u_\perp^2/v_e^2} J_0(au_\perp) du_\perp = \frac{1}{2\pi} \exp \left\{ -\frac{v_e^2 a^2}{4} \right\}. \quad (32)$$

The application of (32) yields

$$\begin{aligned} \int_0^\infty u_\perp du_\perp \int_0^{2\pi} d\varphi \tilde{f}_1(\mathbf{k}, \mathbf{u}, t) &= -\frac{2e}{v_e^2 m_e} \int_0^t d\tau \tilde{\phi}(\mathbf{k}, t - \tau) f_0(u_\parallel) e^{-i\mathbf{k} \cdot \mathbf{V}\tau} \\ &\times \frac{\partial}{\partial \tau} \left\{ \exp \left(-\frac{k_\perp^2 v_e^2}{\Omega_e^2} \sin^2 \frac{\Omega_e \tau}{2} \right) e^{-ik_\parallel u_\parallel \tau} \right\}. \end{aligned} \quad (33)$$

And the integration with respect to u_\parallel now gives

$$\begin{aligned} \int_0^\infty u_\perp du_\perp \int_{-\infty}^\infty du_\parallel \int_0^{2\pi} d\varphi \tilde{f}_1(\mathbf{k}, \mathbf{u}, t) &= -\frac{2e}{v_e^2 m_e} \int_0^t d\tau \tilde{\phi}(\mathbf{k}, t - \tau) e^{-i\mathbf{k} \cdot \mathbf{V}\tau} \\ &\times \frac{\partial}{\partial \tau} \exp \left\{ -\frac{k_\perp^2 v_e^2}{\Omega_e^2} \sin^2 \left(\frac{\Omega_e \tau}{2} \right) - \left(\frac{k_\parallel v_e \tau}{2} \right)^2 \right\}. \end{aligned} \quad (34)$$

Differentiation with respect to τ yields the final result

$$\int \tilde{f}_1(\mathbf{k}, \mathbf{u}, t) d\mathbf{u} = \frac{k^2 e}{m_e} \int_0^t d\tau \Gamma(\mathbf{k}, \tau) \tilde{\phi}(\mathbf{k}, t - \tau), \quad (35)$$

where $\Gamma(\mathbf{k}, \tau)$ is the kernel (9). Substitution of this result into Poisson's equation (7) yields the integral equation (8). This completes the derivation of (8) and (9).

VI. INTEGRAL EQUATION FOR MULTI-SPECIES PLASMAS

Different types of ions and other charged particle species can be taken into account in a straightforward manner. The Vlasov equation for particle species s takes the form

$$\frac{\partial \tilde{f}_{1s}}{\partial t} + i\mathbf{k} \cdot (\mathbf{u} + \mathbf{V}_s) \tilde{f}_{1s} - \Omega_s \frac{\partial \tilde{f}_{1s}}{\partial \varphi} = -\frac{2q_s \tilde{\phi}}{v_s^2 m_s} i\mathbf{k} \cdot \mathbf{u} f_{0s}(u), \quad (36)$$

where $\tilde{f}_{1s}(\mathbf{k}, \mathbf{u}, t)$ is the Fourier transform of $f_{1s}(\mathbf{x}, \mathbf{u}, t)$ and $\Omega_s = q_s B_0 / m_s$ is the signed cyclotron frequency of species s . The effects of the different species are additive in Poisson's equation which becomes

$$k^2 \tilde{\phi}(\mathbf{k}, t) = \sum_s \frac{n_s q_s}{\epsilon_0} \int \tilde{f}_{1s}(\mathbf{k}, \mathbf{u}, t) d\mathbf{u} + \frac{q(t)}{\epsilon_0}. \quad (37)$$

Subject to the initial conditions $\tilde{f}_{1s}(\mathbf{k}, \mathbf{u}, 0) = 0$ and $\tilde{\phi}(\mathbf{k}, 0) = 0$, the Vlasov equation (36) is solved as described in Section IV and the velocity space integrals are computed as in Section V with the result

$$\tilde{\phi}(\mathbf{k}, t) + \int_0^t \Gamma(\mathbf{k}, t - \tau) \tilde{\phi}(\mathbf{k}, \tau) d\tau = \frac{q(t)}{\epsilon_0 k^2}, \quad k \neq 0, \quad (38)$$

where

$$\begin{aligned} \Gamma(\mathbf{k}, \tau) = \sum_s \omega_{ps}^2 \tau \left[\frac{k_{\perp}^2}{k^2} \cdot \frac{\sin(\Omega_s \tau)}{\Omega_s \tau} + \frac{k_{\parallel}^2}{k^2} \right] \\ \times \exp \left\{ -\frac{k_{\perp}^2 v_s^2}{\Omega_s^2} \sin^2 \left(\frac{\Omega_s \tau}{2} \right) - \left(\frac{k_{\parallel} v_s \tau}{2} \right)^2 - i\mathbf{k} \cdot \mathbf{V}_s \tau \right\} \end{aligned} \quad (39)$$

and $\omega_{ps}^2 = n_s q_s^2 / \epsilon_0 m_s$. Even though the integral equation formulation of the Vlasov-Poisson system is exact, the calculation of numerical solutions in multi-species plasmas is difficult in practice because of the disparity between ion and electron timescales.

VII. CONCLUSIONS

It has been shown that the electric field generated by a time varying point charge $q(t)\delta(\mathbf{x})$ in a magnetized plasma can be obtained as the solution of an integral equation and, in the case of a convecting Maxwellian plasma with an isotropic pressure distribution, that the kernel may be expressed in terms of elementary functions. Numerical solutions of this integral equation can be used to study the response of the plasma to various types of forcing $q(t)$ and thus it provides an alternative approach to exact analytic theories and their approximations which are often cumbersome for the analysis of magnetized plasmas. For example, the impulse response of a magnetized plasma can be studied by using pulsed sources such as $q(t) = At \exp(-t/t_0)$ or $q(t) = At \exp(-t^2/2t_0^2)$. These functions both rise to a maximum at $t = t_0$ and then decline to zero; the risetime t_0 is adjustable, of course, for the application of interest. The integral equation can also be used to study wave excitation and propagation for sources with arbitrary waveforms such that $q(t)$ is continuous and satisfies $q(0) = 0$. The integral equation, therefore, should be of practical value.

Concerning the avoidance of infinite series of Bessel functions in studies of homogeneous magnetized plasmas, the recent work by Qin *et al.* [56] should be mentioned. Textbook derivations of the electromagnetic susceptibility of homogeneous, magnetized, gyrotropic plasmas usually employ expansions in infinite series of Bessel functions to perform the integration over the particle

gyro-phase [see, for example, chapter 10 in Ref. 57]. Qin *et al.* [56] have shown how this integration may be carried out without the use of Bessel function expansions. While this is an important mathematical development, the resulting form of the susceptibility tensor derived by Qin *et al.* still contains integrations over the variables p_{\perp} and p_{\parallel} which have not been carried out. In the case of Maxwell or bi-Maxwell velocity distribution functions, the remaining integrations over p_{\perp} and p_{\parallel} inevitably give rise to infinite series of Bessel functions and lead to the well known expressions given, for example, by Stix [57]. Consequently, for Maxwellian plasmas, the expressions for the susceptibility tensor derived by Qin *et al.* have postponed but not eliminated the practical need for such series expansions in numerical calculations of the hot plasma dielectric tensor.

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